Applications of canonical transformations

Subhashish Banerjee and Joachim Kupsch

Fachbereich Physik, Technische Universität Kaiserslautern, D-67653 Kaiserslautern, Germany

E-mail: kupsch@physik.uni-kl.de

Abstract
Canonical transformations are defined and discussed along with the coherent
and the ultracoherent vectors. It is shown that the single-mode and the n-mode
squeezing operators are elements of the group of canonical transformations.
An application of canonical transformations is made, in the context of open
quantum systems, by studying the effect of squeezing of the bath on the
decoherence properties of the system. Two cases are analysed. In the first
case, the bath consists of a massless bosonic field with the bath reference states
being the squeezed vacuum states and squeezed thermal states while in the
second case a system consisting of a harmonic oscillator interacting with a bath
of harmonic oscillators is analysed with the bath being initially in a squeezed
thermal state.

PACS numbers: 02.20.−a, 03.65.Yz, 05.40.−a

1. Introduction

The transformations preserving the form of the Hamilton equations are known as the canonical
transformations in classical mechanics. Their counterparts in quantum mechanics are those that
preserve the commutation relations between the creation and the annihilation operators. Many
physical consequences can be derived from canonical transformations in quantum mechanics
[1]. For instance, in the phase-space picture the uncertainty relations are formulated in terms
of the area occupied by the Wigner function [2]. The spread of the wave packet is an area-
preserving canonical transformation in quantum mechanics. The Lorentz boost in a given
direction is a canonical transformation in phase space using the light-cone variables. This
allows the formulation of the uncertainty relations in a Lorentz-invariant manner.

Decoherence is a consequence of the 'openness' of a physical system [3–8], i.e., the
system is not isolated but is in contact with its environment. This causes the decay of the
quantum interferences between components of quantum superposition states. From a general
point of view but especially for quantum computation it would be very desirable to have some
means of stabilizing the quantum superpositions against action of the environment [9–13].

In this context a number of studies have been made [14, 15], which have shown that the phase-sensitive squeezed reservoirs (environment) are better suited to the task of controlling decoherence than the phase-insensitive thermal heat bath (reservoir). Squeezed baths are characterized by the fact that correlations exist between the bath modes [16].

In this paper, we will discuss some applications of the generalized canonical transformations the details of which have been formulated in [17]. The plan of the paper is as follows. In section 2, we will review canonical transformations illustrating their group structure and discuss the exponential, the coherent and the ultracoherent vectors as well as the canonical transformations in Fock space. The connection between squeezing and canonical transformations [1, 18] has been an active area of research over the years. In section 3, we will demonstrate the single-mode and the n-mode squeezing operators to be elements of the group of canonical transformations and thus apply their unitary ray representations to the coherent as well as the ultracoherent states. In section 4 as an application of squeezing to open quantum systems, we discuss the effect of squeezing of the bath on the decoherence properties of the system. In section 4.1 we discuss the induced superselection rules of a class of Hamiltonian models with the environment given by a massless bosonic field and the environment reference states being squeezed vacuum and squeezed thermal states. In section 4.2, we make use of the squeezed thermal states on a generic model of open quantum systems and demonstrate its ability to put a check on the decoherence properties of the system. In section 5 we make our conclusions. We have also included an appendix in which we discuss some details of the ultracoherent vector and its connection with canonical transformations.

2. Ultracoherent vectors and canonical transformations

In this section, we recapitulate the properties of canonical transformations on a Fock space using coherent and ultracoherent vectors. A more detailed presentation of this technique will be given in [17], see also the appendix of the present paper. We start with some basic definitions and notations about the Fock space of symmetric tensors. Let $\mathcal{H}$ be a Hilbert space with inner product $(f \mid g)$ and with an antiunitary involution $f \rightarrow f^*$. The mapping $f, g \rightarrow \langle f \mid g \rangle := (f^* \mid g) \in \mathbb{C}$ is then a symmetric bilinear form. An explicit and representative example is $\mathcal{H} = \mathbb{C}^n$ with $(f \mid g) = \sum_{\mu=1}^{n} \overline{f}_\mu g_\mu$ and $\langle f \mid g \rangle = \sum_{\mu=1}^{n} f_\mu g_\mu$ for $f = (f_1, \ldots, f_n) \in \mathbb{C}^n$ and $g = (g_1, \ldots, g_n) \in \mathbb{C}^n$. The involution is simply complex conjugation $f^* = (\overline{f}_1, \ldots, \overline{f}_n)$. With the symmetric tensor product $f \circ g$ the Hilbert space $\mathcal{H}$ generates the bosonic Fock space $\mathcal{F}(\mathcal{H})$. The vacuum vector is denoted by 1_{vac}. For all vectors $f \in \mathcal{H}$ the exponential vectors $\exp f = 1_{\text{vac}} + f + \frac{1}{2} f \circ f + \cdots$ converge within $\mathcal{F}(\mathcal{H})$, the inner product being $(\exp f \mid \exp g) = \exp(f \mid g)$. Coherent states are the normalized exponential vectors $\exp\left(f - \frac{1}{2}\|f\|^2\right)$. The linear span of all exponential vectors $\{\exp f \mid f \in \mathcal{H}\}$ is dense in $\mathcal{F}(\mathcal{H})$. To determine an operator on $\mathcal{F}(\mathcal{H})$ it is therefore sufficient to know this operator on all exponential vectors. The involution on $\mathcal{H}$ is naturally extended to an involution on $\mathcal{F}(\mathcal{H})$ such that $(\exp f)^* = \exp f^*$, $f \in \mathcal{H}$. The mapping $f, g \in \mathcal{F}(\mathcal{H}) \rightarrow \langle f \mid g \rangle = (f^* \mid g) \in \mathbb{C}$ is again a bilinear symmetric form.

A class of elements of the Fock space more general than coherent vectors are the ultracoherent vectors [19]. Let A be a Hilbert–Schmidt operator on $\mathcal{H}$, which is symmetric, i.e., $\langle Af \mid g \rangle = \langle f \mid Ag \rangle$ holds for all $f, g \in \mathcal{H}$, then there exists a unique tensor of second degree, in the following denoted by $\Omega(A)$, such that

$$\langle \Omega(A) \mid f \circ g \rangle = \langle Af \mid g \rangle \tag{1}$$

for all $f, g \in \mathcal{H}$. To be more explicit, let $e_\mu = e_\mu^*$ be a real orthonormal basis of the Hilbert space $\mathcal{H}$, then $\Omega(A)$ is the tensor $\Omega(A) = \frac{1}{2} \sum_{\mu\nu} A_{\mu\nu} e_\mu \circ e_\nu$ where the coefficients

$A_{\mu\nu} = \langle e_\mu \mid A e_\nu \rangle = A_{\nu\mu}$ have a convergent sum $\sum_{\mu\nu} |A_{\mu\nu}|^2 < \infty$. If $A^\dagger A < I$ (all eigenvalues strictly less than one) then the exponential series $\exp \Omega(A)$ converges within the Fock space, the norm being $\| \exp \Omega(A) \| = \det(I - A^\dagger A)^{-\frac{1}{4}}$. We denote with $\mathcal{D}_1$ the set of all symmetric Hilbert–Schmidt operators with $A^\dagger A < I$. The convex set $\mathcal{D}_1$ is usually called the Siegel disc. For any operator $Z \in \mathcal{D}_1$ and for any $f \in \mathcal{H}$ we now define the ultracoherent vector

$$\mathcal{E}(Z, f) = \exp \Omega(Z) \circ \exp f = \exp(\Omega(Z) + f) \in \mathcal{F}(\mathcal{H}). \tag{2}$$

The standard building blocks of a bosonic theory are the creation and annihilation operators. Given a vector $f \in \mathcal{H}$ the creation operator $b^\dagger(f)$ of that vector and the annihilation operator $b(f)$ are uniquely determined by

$$b^\dagger(f) \exp g = f \circ \exp g = \frac{\partial}{\partial \lambda} \exp(g + \lambda f)|_{\lambda=0}, \tag{3}$$

$$b(f) \exp g = \langle f \mid g \rangle \exp g = (f^* \mid g) \exp g. \tag{4}$$

These operators satisfy $(b^\dagger(f))^\dagger = b(f^*)$, and they have the commutation relations $[b^\dagger(f), b^\dagger(g)] = [b(f), b(g)] = 0$ and $[b(f), b^\dagger(g)] = \langle f \mid g \rangle$. If we choose an orthonormal basis e_μ, $\mu = 1, 2, \ldots$, of $\mathcal{H}$, then the operators $b^\dagger_\mu = b^\dagger(e_\mu)$ and $b_\nu = b(e_\nu^*)$ form a basis of the operator algebra and satisfy the canonical commutation relations

$$\left[b^\dagger_\mu, b^\dagger_\nu \right] = [b_\mu, b_\nu] = 0, \tag{5}$$

$$\left[b_\mu, b^\dagger_\nu \right] = \delta_{\mu\nu}. \tag{6}$$

Below we shall use a notation of [20]. Let B be a self-adjoint operator on $\mathcal{H}$ and e_μ be a orthonormal basis of the Hilbert space $\mathcal{H}$. Then

$$b^\dagger B b = \sum_{\mu\nu} (e_\mu \mid B e_\nu) b^\dagger_\mu b_\nu \tag{7}$$

is a well-defined self-adjoint operator on the Fock space. The operators $b^\dagger_\mu$, b_ν and the matrix elements $(e_\mu \mid B e_\nu)$ depend explicitly on the choice of the basis, but (7) does not. Let A be a symmetric Hilbert–Schmidt operator then $b^\dagger A b^\dagger$ and $b \overline{A} b$ are defined as

$$b^\dagger A b^\dagger = \sum_{\mu\nu} \langle e_\mu \mid A e_\nu \rangle b^\dagger_\mu b^\dagger_\nu \qquad \text{and} \qquad b \overline{A} b = (b^\dagger A b^\dagger)^\dagger = \sum_{\mu\nu} \overline{\langle e_\mu \mid A e_\nu \rangle} b_\mu b_\nu. \tag{8}$$

For arbitrary elements $h \in \mathcal{H}$ the Weyl operators are defined on the set of exponential vectors by

$$W(h) \exp f = \exp \left(-(h \mid f) - \tfrac{1}{2} \|h\|^2 \right) \exp(f + h). \tag{9}$$

These operators are unitary with $W^\dagger(h) = W(-h) = W^{-1}(h)$. Definition (3) is equivalent to

$$W(h) = \exp(b^\dagger(h) - b(h^*)). \tag{10}$$

Identities (3)–(9) imply

$$W(h) b^\dagger_\mu W(h) = b^\dagger_\mu - \overline{h}_\mu, \tag{11}$$

$$W(h) b_\nu W(h) = b_\nu - h_\nu. \tag{12}$$

Hence the Weyl operators are the displacement operators of quantum optics. As already stated, the linear span of exponential vectors or coherent vectors is dense in the Fock space. From (9) it immediately follows that the Weyl operators map this set into itself. Moreover, the identity

$W(h)1_{\text{vac}} = \exp\left(h - \frac{1}{2}\|h\|^2\right)$ implies that the linear span of $\{W(h)1_{\text{vac}} \mid h \in \mathcal{H}\}$ is exactly the linear span of all coherent states. The action of the Weyl operator on an ultracoherent vector is [17]

$$W(h) \exp(\Omega(A) + f) = e^{-\frac{1}{2}\|h\|^2 + \frac{1}{2}\langle h^* \mid Ah^* - 2f\rangle} \exp(\Omega(A) + f + h - Ah^*). \tag{13}$$

Hence the Weyl operators map ultracoherent vectors onto ultracoherent vectors.

Canonical transformations are affine linear transformations between the creation and annihilation operators preserving the commutation relations. From (11) and (12) it immediately follows that Weyl operators generate inhomogeneous canonical transformations. Treating the most general linear homogeneous transformations we have

$$b^\dagger(f) \to b^\dagger(Uf) - b(\overline{V}f), b(f) \to b(\overline{U}f) - b^\dagger(Vf). \tag{14}$$

Here U and V are bounded operators on the Hilbert space $\mathcal{H}$. The operators $\overline{U}$ (and $\overline{V}$) are defined by $\overline{U}f = (Uf^*)^*$ and U^T means $U^T = (\overline{U})^\dagger = \overline{U^\dagger}$. In the case of $\mathcal{H} = \mathbb{C}^n$ the corresponding matrices are just the complex conjugate or the transposed matrix. Transformation (14) is equivalent to the following transformation of the argument of the Weyl operator (10),

$$b^\dagger(f) - b(f^*) \to b^\dagger(Uf + Vf^*) - b(\overline{U}f^* + \overline{V}f), \tag{15}$$

which can be better visualized by the mapping of the test functions

$$\begin{pmatrix} f \\ f^* \end{pmatrix} \to \begin{pmatrix} Uf + Vf^* \\ \overline{V}f + \overline{U}f^* \end{pmatrix} = \widehat{G}\begin{pmatrix} f \\ f^* \end{pmatrix} \tag{16}$$

where the matrix of operators

$$\widehat{G} = \begin{pmatrix} U & V \\ \overline{V} & \overline{U} \end{pmatrix} \tag{17}$$

maps the underlying real space of $\mathcal{H}$, parametrized by the vectors $\begin{pmatrix} f \\ f^* \end{pmatrix}$, $f \in \mathcal{H}$, into itself. Transformations (14) preserve the canonical commutation relations, if

$$\widehat{G}\Theta\widehat{G}^\dagger = \Theta \tag{18}$$

and

$$\Delta\widehat{G}\Delta = \overline{\widehat{G}} \tag{19}$$

with

$$\Theta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \Delta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{20}$$

From these, we can derive the equivalent conditions for the transformations to be canonical as

$$UU^\dagger - VV^\dagger = I, \qquad UV^T = VU^T, \tag{21}$$

or

$$U^\dagger U - V^T\overline{V} = I, \qquad U^T\overline{V} = V^\dagger U. \tag{22}$$

Operators (17) form the group $\mathcal{G}_c$ of linear canonical transformations, which are often called Bogoliubov transformations [21]. Thereby, it is sufficient to identify the mapping $G = G(U, V)$ in the first line of (16)

$$f \in \mathcal{H} \to G(U, V)f = Uf + Vf^* \in \mathcal{H}, \tag{23}$$

which is an $\mathbb{R}$-linear transformation on $\mathcal{H}$. The successive application of canonical transformations corresponds to the multiplication of the respective matrix operators (17)

or of the respective operators (23). In the latter case the multiplication law follows from the definition as

$$G_2 G_1 f = G_2(U_1 f + V_1 f^*) = (U_2 U_1 + V_2 \overline{V}_1) f + (U_2 V_1 + V_2 \overline{U}_1) f^*. \quad (24)$$

The inverse mapping of (23) is

$$G^{-1}(U, V) f = U^\dagger f - V^T f^* = G(U^\dagger, -V^T) f. \quad (25)$$

In the finite-dimensional case $\mathcal{H} = \mathbb{C}^n$ identity (18) implies that $\mathcal{G}_c$ is a subgroup of $SU(n, n)$. Identity (19) is an additional reality constraint, such that $\mathcal{G}_c$ is isomorphic to the real symplectic group [22].

For finite-dimensional Hilbert spaces $\mathcal{H}$ the canonical transformations (14) can always be implemented by unitary operators on the Fock space $\mathcal{F}(\mathcal{H})$; in the infinite-dimensional case one needs the additional constraint that V is a Hilbert–Schmidt operator [20, 23]. In [20] these transformations are called proper canonical transformations. As easily seen the subclass of proper canonical transformations forms a group. In the subsequent investigations the group $\mathcal{G}_c$ is the group of all proper canonical transformations.

In order to define canonical transformations in Fock space, we set up a projective representation of the group $\mathcal{G}_c$ by identifying for each element G of $\mathcal{G}_c$ a unitary operator $T(G)$ on $\mathcal{F}(\mathcal{H})$ such that

$$T(id) = I, T(G_2)T(G_1) = \omega(G_2, G_1)T(G_2 G_1) \quad (26)$$

with a multiplier $\omega(G_2, G_1) \in \mathbb{C}$, $|\omega(G_2, G_1)| = 1$. It is sufficient to define $T(G)$ on the set of exponential vectors

$$T(G) \exp f = \det |U|^{-\frac{1}{2}} \exp \left(\Omega(U^{\dagger-1} V^T) + U^{\dagger-1} f - \tfrac{1}{2}\langle f \mid V^\dagger U^{\dagger-1} f \rangle \right). \quad (27)$$

Thereby the operator $|U| = \sqrt{UU^\dagger} = \sqrt{I + VV^\dagger} \geqslant I$ is the positive self-adjoint part of U. Since V is a Hilbert–Schmidt operator, we know that $|U| - I$ is a trace class operator, and the determinant $\det |U|$ is well defined also if $\dim \mathcal{H} = \infty$. Formula (27) shows that in general canonical transformations map coherent vectors—including the vacuum—onto ultracoherent vectors. This class of vectors turns out to be stable against canonical transformations, see the appendix and [17]. In order to work out the group structure of $T(G)$, its action on the ultracoherent vector (2) is also needed. This is given in the appendix and [17].

If $\dim \mathcal{H} < \infty$ we can define a representation (27) with the determinant of U (instead of $|U|$). Then the multiplier ω in (26) takes only the values $\omega = \pm 1$, in agreement with the result of Bargmann [22], who has investigated the finite-dimensional case. In the infinite-dimensional case $\det U$ is not defined for all (proper) canonical transformations, and we have to choose $\det |U|$. But then ω can take any value on the unit circle. The group $\mathcal{G}_c$ is still a topological group [23], but a Lie algebra can only be defined for a subclass of canonical transformations with bounded generators, see chapter 3 of [24].

All (proper) canonical transformations are products of the following two classes of canonical transformations.

(i) Take a (bounded or unbounded) self-adjoint operator $\Psi = \Psi^\dagger$ on $\mathcal{H}$. Then

$$U = \exp i\Psi \quad (28)$$

is a unitary operator, and U and $V = 0$ obviously satisfy conditions (21). Transformation (23) $G(U, 0) = U$ is therefore a proper canonical transformation. The representation of such a unitary operator is $T(G(U, 0))$ which we simply denote by $R(U)$. From (27) we obtain

$$R(U) \exp f = \exp Uf. \quad (29)$$

In this case the homogeneous canonical transformations map coherent states onto coherent states, and ultracoherent vectors are mapped onto

$$R(U)\exp(\Omega(Z)+f)=\exp(\Omega(UZU^T)+Uf). \tag{30}$$

(ii) As the second case of a canonical transformation take a symmetric Hilbert–Schmidt operator $\Xi=\Xi^T$ on $\mathcal{H}$. Then

$$\widehat{G}=\exp\begin{pmatrix}0&\Xi\\\overline{\Xi}&0\end{pmatrix}=\begin{pmatrix}U&V\\\overline{V}&\overline{U}\end{pmatrix} \tag{31}$$

is a matrix operator of type (17) with the bounded operators

$$U=\cosh\sqrt{\Xi\overline{\Xi}}\geqslant I\qquad\text{and}\qquad V=\Xi\frac{\sinh\sqrt{\overline{\Xi}\,\Xi}}{\sqrt{\overline{\Xi}\,\Xi}}=\frac{\sinh\sqrt{\Xi\overline{\Xi}}}{\sqrt{\Xi\overline{\Xi}}}\Xi=V^T, \tag{32}$$

which satisfy conditions (21). Moreover, $U-I$ is a positive trace class operator and V is a Hilbert–Schmidt operator. Operators (31) are exactly those operators within the group $\mathcal{G}_c$, which are positive, see [22, 23]. We use the short notation $G_\Xi=G\left(\cosh\sqrt{\Xi\overline{\Xi}},\Xi(\overline{\Xi}\,\Xi)^{-\frac{1}{2}}\sinh\sqrt{\overline{\Xi}\,\Xi}\right)$ for the group elements, and $S(\Xi)=T(G_\Xi)$ for the representation within the Fock space. Definition (27) yields

$$S(\Xi)\exp f=\det(\cosh\sqrt{\Xi\overline{\Xi}})^{-\frac{1}{2}}\exp\Bigg\{\Omega\left(\frac{\tanh\sqrt{\Xi\overline{\Xi}}}{\sqrt{\Xi\overline{\Xi}}}\Xi\right)$$

$$+(\cosh\sqrt{\Xi\overline{\Xi}})^{-1}f-\frac{1}{2}\left\langle f\,\Big|\,\overline{\Xi}\frac{\tanh\sqrt{\Xi\overline{\Xi}}}{\sqrt{\Xi\overline{\Xi}}}f\right\rangle\Bigg\}. \tag{33}$$

In the next section we shall see that these operators produce the squeezing of quantum optics.

If $\Xi=\overline{\Xi}=\Xi^\dagger$ is real and self-adjoint, then (31) becomes

$$\widehat{G}=\begin{pmatrix}\cosh\Xi&\sinh\Xi\\\sinh\Xi&\cosh\Xi\end{pmatrix} \tag{34}$$

and (33) simplifies to

$$S(\Xi)\exp f=\det(\cosh\Xi)^{-\frac{1}{2}}\exp(\Omega(\tanh\Xi)+(\cosh\Xi)^{-1}f-\tfrac{1}{2}\langle f\,|\,\tanh\Xi f\rangle). \tag{35}$$

In [22, 23] it has been shown that any group element $G_0\in\mathcal{G}_c$ can be written as a product

$$G_0=G(U_1,0)G(U_2,V_2), \tag{36}$$

where U_1 is unitary and the operators U_2 and V_2 are operators of type (32). Factorization (36) is actually the polar decomposition of the bounded operator G_0 into a unitary operator and a positive one. To investigate canonical transformations, it is therefore sufficient to consider the two classes of operators given above.

From (28) and (31) we see that these canonical transformations can be considered as elements of one-parameter subgroups, and we can easily obtain the generators of these groups.

The unitary operator (28) can be extended to a one-parameter group $U(t)=\exp i\Psi t$. The (possibly unbounded) generator of the group $T(U(t))$ is then calculated from $K_\Psi\exp f:=-i\frac{d}{dt}T(\exp it\Psi)\exp f|_{t=0}=\Psi f\circ\exp f$ as

$$K_\Psi=b^\dagger\Psi b, \tag{37}$$

such that the operator $R(U)$ (29) is given by

$$R=\exp ib^\dagger\Psi b. \tag{38}$$

For the proof of identity (37) it is sufficient to choose a rank-one operator $\Psi = |g\rangle\langle g^*|$. Then $b^\dagger \Psi b = b^\dagger(g) b(g^*)$ and $b^\dagger(g) b(g^*) \exp f = g \langle g^* | f \rangle \exp f = \Psi f \circ \exp f$.

For fixed Ξ the operators $G(\lambda) = G_{\lambda\Xi}$, $\lambda \in \mathbb{R}$, form a one-parameter group of symplectic transformations with $G(0) = id$ and $G(\lambda_1) G(\lambda_2) = G(\lambda_1 + \lambda_2)$. Using formula (A.5) it is straightforward to check that $G(\lambda) \rightarrow S(\lambda) = T(G(\lambda))$ is a faithful unitary representation of this subgroup. If $S(\lambda)$ is applied to coherent states we obtain from (33)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} S(\lambda) \exp f|_{\lambda=0} = \Omega(\Xi) \circ \exp f - \frac{1}{2} \langle f | \overline{\Xi} f \rangle \exp f = K_\Xi \exp f. \tag{39}$$

Since the linear span of coherent states is dense this completely fixes the generator of this group. Using creation and annihilation operators this generator is identified with

$$K_\Xi = \tfrac{1}{2}(b^\dagger \Xi b^\dagger - b \overline{\Xi} b), \tag{40}$$

and S_Ξ is given by

$$S(\Xi) = \exp \tfrac{1}{2}(b^\dagger \Xi b^\dagger - b \overline{\Xi} b). \tag{41}$$

To prove identity (40) it is sufficient to choose a symmetric rank-one operator $\Xi = |h\rangle\langle h|$ with $h \in \mathcal{H}$. Then the application of $b^\dagger \Xi b^\dagger - b \overline{\Xi} b = b^\dagger(h) b^\dagger(h) - b(h^*) b(h^*)$ onto a coherent state yields, see (3) and (4), $(b^\dagger \Xi b^\dagger - b \overline{\Xi} b) \exp f = (h \circ h - \langle h^* | f \rangle^2) \exp f$. On the other hand we have $2\Omega(\Xi) = h \circ h$ and $\langle f | \overline{\Xi} f \rangle = \langle h^* | f \rangle^2$, and (40) follows.

The Weyl operator and the homogeneous canonical transformations are related by the identity

$$T(G) W(h) T^+(G) = W(Gh), \tag{42}$$

where G is the mapping (23). This follows from (13) and (A.5). Actually this identity can already be inferred from (10) and (15), if the existence of the unitary representation $T(G)$ is taken for granted.

3. Squeezed states and connection with canonical transformations

3.1. Single-mode squeeze operator

If $\mathcal{H} = \mathbb{C}$ the expressions of section 2 simplify considerably. Then the operator Ξ in (31) is just a complex number $\xi \in \mathbb{C}$. The single-mode squeeze operator is now defined in agreement with the squeeze operators in [25, 26] as a special case of (41)

$$S_\xi = \exp\left(\tfrac{1}{2}(\xi b^{\dagger 2} - \xi^* b^2)\right) = \exp(\mathrm{i} O(\xi)) \tag{43}$$

where $\xi = r \mathrm{e}^{\mathrm{i}\theta}$ and

$$O(\xi) = \frac{1}{2\mathrm{i}}(\xi b^{\dagger 2} - \xi^* b^2). \tag{44}$$

Here b, $b^\dagger$ are the creation and annihilation operators. Now using a single-mode rotation operator defined as, see (38),

$$T(\phi) = \exp(\mathrm{i}\phi b^\dagger b) \qquad \text{with} \quad \phi \in \mathbb{R} \tag{45}$$

one has

$$T^\dagger(\phi) O(\xi) T(\phi) = O(\xi \mathrm{e}^{-\mathrm{i}2\phi}). \tag{46}$$

For $\phi = \frac{\theta}{2}$ it follows from the above equation that $O(\xi)$ is unitarily equivalent to $O(r)$. Thus we can see that S_ξ is unitarily equivalent to S_r. In the notation of section 2 the operator S_r is exactly of the form (35).

The Lie algebra of the squeeze operator defined by equation (43) is spanned by

$$B_+ = \tfrac{1}{2} b^\dagger b^\dagger, \qquad B_- = -\tfrac{1}{2} bb, \qquad J_3 = \tfrac{1}{2}\left(b^\dagger b + \tfrac{1}{2}\right). \tag{47}$$

These operators satisfy the commutation relations [27]

$$[B_+, B_-] = 2J_3, \qquad [J_3, B_\pm] = \pm B_\pm. \tag{48}$$

3.2. n-mode squeeze operator

If $\mathcal{H} = \mathbb{C}^n$ we can generalize this case to n-modes. Let Ξ be an $n \times n$ symmetric (complex) matrix. We define the n-mode squeeze operator as the canonical transformation (41)

$$S(\Xi) = \exp \tfrac{1}{2}(b^\dagger \Xi b^\dagger - b\overline{\Xi}b). \tag{49}$$

An n-mode rotation operator is the canonical transformation (38)

$$T(\Phi) = \exp(ib^\dagger \Phi b) \tag{50}$$

where Φ is an $n \times n$ Hermitian matrix. From (30) and (33) the well-known identity

$$T^\dagger(\Phi)S(\Xi)T(\Phi) = S\left(e^{-i\Phi}\,\Xi\,e^{-i\Phi^T}\right), \tag{51}$$

cf [25], follows. Using the fact that Ξ is an $n \times n$ symmetric matrix and Φ is an $n \times n$ Hermitian matrix, it can be shown (cf [28]: appendix II, lemma 1) that

$$e^{-i\hat{\Phi}}\,\Xi\,e^{-i\hat{\Phi}^T} = \Xi_D \tag{52}$$

where Ξ_D is real diagonal with non-negative elements $(d_1, \ldots, d_n)$. Thus S_Ξ is unitarily equivalent to

$$T^\dagger(\Phi)S(\Xi)T(\Phi) = S^{(1)}(d_1)S^{(2)}(d_2) \cdots S^{(n)}(d_n) \tag{53}$$

where $S^{(k)}$ denotes a single-mode (k) squeeze operator as considered above. The action of the squeezing operator on the exponential and the ultracoherent vectors can be deduced from equations (35) and (A.5), respectively.

4. Effect of squeezing of the bath on the decoherence properties of the system

In this section, we will discuss the effect of squeezing of the bath on the decoherence properties of the system. In section 4.1, we take up a class of models with a massless bosonic field representing the environment (bath) with the squeezed vacuum and squeezed thermal states representing the reference states of the environment and in section 4.2 we make use of the squeezed thermal states on a generic open quantum system model and demonstrate its ability to put a check on the decoherence properties of the system.

4.1. Superselection and squeezing

The central idea behind 'open systems' is that a system is not isolated but in contact with its environment (reservoir) which influences the time evolution of the system making it nonunitary. Thereby classical properties can arise due to superselection rules induced by the environment [5]. The central feature in these studies is the reduced density matrix of the system (ρ_S) of interest obtained by taking a trace over the environment

$$\rho_S(t) = \mathrm{tr}_R U(t)(\rho_S \otimes \rho_R)U^\dagger(t) \tag{54}$$

where $\rho(0) = \rho_S \otimes \rho_R$ is the initial state of the system–reservoir complex and $U(t)$ is the unitary operator describing the unitary time evolution of the entire system–reservoir complex. Here S and R stand for the system and reservoir, respectively.

The dynamics of the total system–reservoir complex is said to induce superselection rules [29, 30] into the system S, if there exist projection operators $\{P_S(\Delta) \mid \Delta \subset \mathbb{R}\}$ on the Hilbert space $\mathcal{H}_S$ such that

$$P_S(\Delta^1)\rho_S(t)P_S(\Delta^2) \to 0 \qquad \text{if} \quad t \to \infty \text{ and } \text{dist}\,(\Delta^1, \Delta^2) > 0, \tag{55}$$

i.e., the off-diagonal parts $P_S(\Delta^1)\rho_S(t)P_S(\Delta^2)$ of the statistical operators of the system S are dynamically suppressed. In any concrete case one has to specify this decrease. For our model we can derive a uniform decrease of the trace norm

$$\|P_S(\Delta^1)\rho_S(t)P_S(\Delta^2)\|_1 \to 0 \qquad \text{if} \quad t \to \infty \text{ and } \text{dist}(\Delta^1, \Delta^2) > 0. \tag{56}$$

Here the projection operators $P_S(\Delta)$ are defined for all intervals $\Delta \subset \mathbb{R}$ of the real line and satisfy

$$\begin{aligned}
P_S(\Delta^1 \cup \Delta^2) &= P_S(\Delta^1) + P_S(\Delta^2) && \text{if} \quad \Delta^1 \cap \Delta^2 = \emptyset \\
P_S(\Delta^1)P_S(\Delta^2) &= P_S(\Delta^1 \cap \Delta^2), && P_S(\emptyset) = 0, \quad P_S(\mathbb{R}) = 1.
\end{aligned} \tag{57}$$

Now we take a model with the Hamiltonian, see [29, 30] for details,

$$H = H_S \otimes I_R + I_S \otimes H_R + V_S \otimes V_R, \tag{58}$$

with

$$H_S = \tfrac{1}{2}P^2 \qquad \text{and} \qquad V_S = P \tag{59}$$

where $P = -\mathrm{id}/\mathrm{d}x$ is the momentum operator of the particle. We thus have a velocity coupling and a massless boson field is taken as the reservoir. Here H_S is the positive Hamiltonian of S, H_R is the positive Hamiltonian of R, and $V_S \otimes V_R$ is the interaction potential between S and R with operators V_S on $\mathcal{H}_S$ and V_R on $\mathcal{H}_R$. The unitary operator giving the total system–reservoir dynamics is given by

$$U(t) = (U_S(t) \otimes I_R) \int P_S(\mathrm{d}\lambda) \otimes \exp(-\mathrm{i}(H_R + \lambda V_R)t), \tag{60}$$

where $U_S(t) = \exp(-\mathrm{i}H_S t)$, and $P_S(\Delta)$, $\Delta \subset \mathbb{R}$, is the family of projection operators coming from the spectral resolution of V_S

$$V_S = \int_{\mathbb{R}} \lambda P_S(\mathrm{d}\lambda). \tag{61}$$

Exactly these projection operators generate the superselection sectors of the model.

The reduced density matrix of the system given by equation (54) becomes

$$\rho_S(t) = U_S(t) \left(\int_{\mathbb{R} \times \mathbb{R}} \chi(\alpha, \beta; t) P_S(\mathrm{d}\alpha)\rho_S P_S(\mathrm{d}\beta) \right) U_S^\dagger(t), \tag{62}$$

with the trace over the reservoir

$$\chi(\alpha, \beta; t) = \text{tr}_R(\mathrm{e}^{\mathrm{i}(H_R + \alpha V_R)t}\mathrm{e}^{-\mathrm{i}(H_R + \beta V_R)t}\rho_R). \tag{63}$$

As a concrete case we take a massless boson field as reservoir. The Hilbert space $\mathcal{H}_R$ is the Fock space $\mathcal{F}(\mathcal{H}_1)$ generated by the one-particle space $\mathcal{H}_1$ of the bosons. The Hamiltonian H_R is given by

$$H_R = \int \mathrm{d}^n k\,\varepsilon(k) a_k^\dagger a_k \tag{64}$$

where $\varepsilon(k) = c|k|$ ($c > 0, k \in \mathbb{R}^n$) is the positive energy function associated with the one-particle Hamilton operator M on $\mathcal{H}_1$

$$(Mf)(k) = \varepsilon(k) f(k). \tag{65}$$

The interaction potential V_R is taken here as the self-adjoint operator

$$V_R = \Phi(h) := a^\dagger(h) + a(h), \tag{66}$$

where the real vector $h = h^* \in \mathcal{H}_1$ satisfies the constraint $2\|M^{-\frac{1}{2}}h\| \leqslant 1$. This enables us to define the Hamiltonian given by equation (58) as a well-defined semibounded operator. The Hamiltonian $H(\alpha) = H_R + \alpha\Phi(h)$ describes the van Hove model [31]. It is defined as a semibounded self-adjoint operator on the Fock space $\mathcal{F}(\mathcal{H}_1)$ if $h \in \mathcal{D}(M^{-\frac{1}{2}}) \subset \mathcal{H}_1$, i.e., h is in the domain of $M^{-\frac{1}{2}}$ where M is the one-particle Hamilton operator of the boson field.

If the reference state of the environment is a coherent state, then the trace (63) coincides up to a phase factor with the expectation of a Weyl operator in the state of the environment [29, 30]

$$\chi(\alpha, \beta; t) = e^{-i\varphi(\alpha,\beta,t)} \text{tr}_R W((\alpha - \beta)k(t))\rho_R \tag{67}$$

with the vector

$$k(t) = (e^{iMt} - I)M^{-1}h = M^{-1}(\cos Mt - I)h + iM^{-1}\sin Mth. \tag{68}$$

The phase $\varphi(\alpha, \beta, t)$, which also depends on the reference state, is not needed for the following arguments. The trace in (67) is easily calculated if the reference state ρ_R is the vacuum. Making use of the fact that the Weyl operator acts on the vacuum to produce the coherent state and the expectation of the Weyl operator in the vacuum is given by

$$(1_{\text{vac}} \mid W(h)1_{\text{vac}}) = e^{-\frac{1}{2}\|h\|^2}, \tag{69}$$

as can be inferred from equation (A.7) in the appendix by setting $f = g = 0$, the trace in (67) follows as

$$\chi(\alpha, \beta; t) = e^{-i\varphi(\alpha,\beta,t)} \exp\left\{-\tfrac{1}{2}(\alpha - \beta)^2\|k(t)\|^2\right\}. \tag{70}$$

Here $k(t)$ is as given in equation (68). In [29, 30] it has been shown that operator (61) is a superselection operator, if

$$\|k(t)\|^2 = \|(I - \cos Mt)M^{-1}h\|^2 + \|M^{-1}\sin Mth\|^2 \tag{71}$$

diverges for $t \to \infty$, and it is found that the conditions for such a divergence are $h \in \mathcal{D}(M^{-\frac{1}{2}})$ and $h \notin \mathcal{D}(M^{-1})$. These conditions also require that the boson field becomes infrared divergent [32, 33], i.e., the boson field is still defined on the Fock space, but the bare boson number diverges and the ground state disappears into the continuum.

This result was obtained for the vacuum as reference state. To investigate the model with a squeezed vacuum as reference state, we choose a symmetric Hilbert–Schmidt operator Ξ on $\mathcal{H}_1$. The operator $S(\Xi)$ (33) generates the squeezed vacuum state

$$1_\Xi = S(\Xi)1_{\text{vac}} \in \mathcal{F}(\mathcal{H}_1). \tag{72}$$

Now we make use of identity (42) with the canonical transformation $G = G(\cosh \Xi, -\sinh \Xi) = G^{-1}(\cosh \Xi, \sinh \Xi)$. Then the trace in (67) follows from

$$(1_\Xi \mid W(\tilde{k})1_\Xi) = (1_{\text{vac}} \mid S^\dagger(\Xi)W(\tilde{k})S(\Xi)1_{\text{vac}}) = (1_{\text{vac}} \mid W(G\tilde{k})1_{\text{vac}})$$
$$= \exp\left(-\tfrac{1}{2}\|G\tilde{k}\|^2\right)$$

where $\tilde{k} = (\alpha - \beta)k(t)$ with $k(t)$ as in equation (68). The condition for induced superselection rules therefore depends on the divergence of

$$\|G\tilde{k}(t)\|^2 = (\alpha - \beta)^2\|(\cosh \Xi)k(t) - (\sinh \Xi)k^*(t)\|^2$$
$$= (\alpha - \beta)^2\|k(t) + (\cosh \Xi - I)k(t) - (\sinh \Xi)k^*(t)\|^2. \tag{73}$$

Since the mapping G is bounded and has a bounded inverse, this norm diverges exactly under the same conditions as (71) does. A closer inspection shows that the leading divergent

contribution comes from $k(t)$, which coincides with (68). The operators $\cosh \Xi - I$ and $\sinh \Xi$ are Hilbert–Schmidt operators and the terms $(\cosh \Xi - I)k(t)$ and $(\sinh \Xi)k^*(t)$ may substantially contribute at intermediate times, but the asymptotics for large times is dominated by $k(t)$.

In [30] also the case of a bath with inverse temperature $\beta > 0$ has been considered. Then instead of the vacuum expectation (69) we need the expectation of the Weyl operator in a thermal state. For the boson system with the one-particle Hamiltonian M this expectation is the Gaussian function

$$\langle W(h) \rangle_\beta = \exp\left(-\left(h\middle|\left((e^{\beta M} - I)^{-1} + \tfrac{1}{2}\right)h\right)\right),$$

which is always smaller than the vacuum expectation, $\langle W(h)\rangle_\beta < \exp\left(-\tfrac{1}{2}\|h\|^2\right)$. Hence superselection sectors are induced even faster than at temperature zero, if (71) diverges. In a squeezed temperature state the expectation of the Weyl operator is given by

$$\langle W(h)\rangle_{\beta,\Xi} = \langle W(Gh)\rangle_\beta, \tag{74}$$

where G is again the canonical transformation $G = G(\cosh \Xi, -\sinh \Xi)$. As in the case of the vacuum, induced superselection sectors follow from the divergence of (73). Thereby the decoherence can be strongly influenced by squeezing at intermediate times, but the behaviour at large times is the same as for the unsqueezed temperature state.

Thus we see that for this model, the squeezing of the bath does not put any check on the superselection properties of the system.

4.2. Open quantum system with a squeezed thermal bath

We take the model Hamiltonian

$$H = \frac{1}{2}M[\dot{x}^2 + \Omega^2 x^2] + \sum_{n=1}^{N}\left[\frac{1}{2}m_n \dot{q}_n^2 + \frac{1}{2}m_n \omega_n^2 \left(q_n - \frac{c_n x}{m_n \omega_n^2}\right)^2\right]. \tag{75}$$

This Hamiltonian can be partitioned as

$$H = H_S + H_R + H_{SR}, \tag{76}$$

where

$$H_S = \tfrac{1}{2}M[\dot{x}^2 + \Omega^2 x^2] \tag{77}$$

is the system Hamiltonian,

$$H_R = \sum_{n=1}^{N}\frac{1}{2}m_n\left[\dot{q}_n^2 + \omega_n^2 q_n^2\right] \tag{78}$$

is the reservoir Hamiltonian and

$$H_{SR} = -\sum_{n=1}^{N}[c_n x q_n] + x^2 \sum_{n=1}^{N}\frac{c_n^2}{2m_n \omega_n^2} \tag{79}$$

is the system–reservoir interaction Hamiltonian. Here the last term in equation (79) compensates for coupling-induced renormalization of the potential. Hamiltonian (75) can be used to describe, for example, the case of a linearly dissipative vibrational degree of freedom, which can be used as a quantum bus in linear ion trap quantum computers [12]. The system Hamiltonian (77) then describes a collective vibrational mode of a linear chain of trapped ions with frequency Ω. We use separable initial conditions, i.e., the system and

reservoir are initially uncorrelated with the initial state of the reservoir being a squeezed thermal initial state

$$\rho_R(0) = S\rho_{\text{th}}S^\dagger. \tag{80}$$

Here

$$\rho_{\text{th}} = \left[1 - \exp\left(\frac{-\hbar\omega}{k_BT}\right)\right]\sum_n \exp\left(\frac{-n\hbar\omega}{k_BT}\right)|n\rangle\langle n| \tag{81}$$

is a thermal density matrix at temperature $T = \beta^{-1}$ and

$$S = S(\Xi) \tag{82}$$

is a squeeze operator of section 3 (see also [26, 34]). This definition of a squeezed thermal bath exactly corresponds to that of section 4.1; the expectation of the Weyl operator in state (81) has the form (74).

By taking the trace over the environment degrees of freedom, we obtain the master equation for the system of interest [35] from which we can get the Wigner equation by the following prescription [2, 36]:

$$\frac{\partial}{\partial t}W(p, x, t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}y\, \mathrm{e}^{\frac{i}{\hbar}py}\left\langle x - \frac{1}{2}y\left|\frac{\partial}{\partial t}\rho_S\right|x + \frac{1}{2}y\right\rangle. \tag{83}$$

The trace operation, to get the reduced density matrix, involved the expectation of the Weyl operator in the thermal state. This reveals the intimate connection of canonical transformations with open system dynamics. The connection between the Wigner function and the Weyl operator is illustrated by the following relation,

$$C(\alpha, \beta) = \text{tr}(\rho\, \mathrm{e}^{i(\alpha\hat{x}+\beta\hat{p})}) = \int\mathrm{d}x\int\mathrm{d}p\, \mathrm{e}^{i(\alpha x+\beta p)}W(x, p) \tag{84}$$

where $\mathrm{e}^{i(\alpha\hat{x}+\beta\hat{p})}$ is the canonical form of the Weyl operator and $W(x, p)$ is the Wigner function. The inverse of this function gives the Wigner function as a function of the trace as

$$W(x, p) = \frac{1}{(2\pi)^2}\int\mathrm{d}\alpha\int\mathrm{d}\beta\, \mathrm{e}^{-i(\alpha x+\beta p)}C(\alpha, \beta). \tag{85}$$

Thus the Wigner equation for the system of interest is obtained as

$$\frac{\partial W}{\partial t} = -\frac{1}{M}\frac{\partial}{\partial x}pW + M\Omega_{\text{ren}}^2(t)\frac{\partial}{\partial p}xW + 2\Gamma(t)\frac{\partial}{\partial p}pW - \hbar D_{pp}(t)\frac{\partial^2}{\partial p^2}W$$
$$- \hbar(D_{xp}(t) + D_{px}(t))\frac{\partial^2}{\partial x\partial p}W - \hbar D_{xx}(t)\frac{\partial^2}{\partial x^2}W. \tag{86}$$

Up to this point our treatment has been exact and is valid for any reservoir spectral density. The general expressions for the coefficients of the Wigner equation (86) can be found in [35]. Now, for the simplicity of computations, we take an Ohmic reservoir with the spectral density

$$I(\omega) = \frac{2}{\pi}\gamma_0 M\omega. \tag{87}$$

In the high-temperature limit we can obtain the Wigner equation coefficients as [35]

$$\Omega_{\text{ren}}^2(t) = \Omega^2 + \frac{2}{\pi}\gamma_0\overline{\Omega}, \tag{88}$$

$$\Gamma(t) = \frac{p}{2}, \tag{89}$$

$$D_{xx}(t) = \frac{2k_B T \gamma_0}{\hbar M \zeta^2} \overline{K}_2 \, e^{-p(t-a)} \sin(\zeta t) \sin[\zeta(t - 2a)], \tag{90}$$

$$D_{xp}(t) = D_{px}(t) = \frac{2k_B T \gamma_0}{\hbar \zeta^2} \left[\zeta \cot(\zeta t) - \frac{p}{2} \right] \overline{K}_2 \, e^{-p(t-a)} \sin(\zeta t) \sin[\zeta(t - 2a)], \tag{91}$$

$$D_{pp}(t) = -\frac{2Mk_B T \gamma_0}{\hbar} \left[K_1 - \overline{K}_2 \, e^{-p(t-a)} \left\{ \left[\cos^2(\zeta t) + \frac{p^2}{4\zeta^2} \sin^2(\zeta t) \right.\right.\right.$$

$$\left.\left.\left. - \frac{p}{2\zeta} \sin(2\zeta t) \right] - 1 \right\} \frac{\sin[\zeta(t - 2a)]}{\sin(\zeta t)} \right]. \tag{92}$$

Here

$$p = 4\gamma_0, \tag{93}$$

$$\zeta = \left(\Omega_{\text{ren}}^2 - \frac{p^2}{4} \right)^{1/2}, \tag{94}$$

$$K_1 = \cosh(2r(\omega)) = \cosh(2r), \tag{95}$$

$$\overline{K}_2 = \sinh(2r(\omega)) = \sinh(2r), \tag{96}$$

$$\theta(\omega) = a\omega, \tag{97}$$

where a is a constant depending upon the squeezing parameters and $\overline{\Omega}$ is the cut-off frequency of the Ohmic reservoir. Here r and θ refer to the amplitude and the phase parts respectively of the complex term in the squeezing operator (cf ξ in equation (43)). The case where there is no squeezing can be obtained from the above equations by setting K_1 to one and $\overline{K}_2$ and a to zero.

In the Wigner equation coefficients given by equations (89)–(92), Γ denotes the term generating dissipation, D_{xx} is responsible for diffusion in p^2, D_{xp} and D_{px} (called the anomalous diffusion terms) generate diffusion in $xp + px$ while D_{pp} is the term responsible for decoherence in x.

It can be seen from the above expressions that for the case of phase insensitive thermal reservoirs we recover the usual high-T results, i.e., the decoherence generating term D_{pp} is a constant proportional to the temperature while the dissipation generating term Γ is equal to $2\gamma_0$. The other terms D_{xx} (diffusion in p^2) and D_{xp}, D_{px} (diffusion in $xp + px$) are zero.

For the case of phase-sensitive squeezed thermal reservoir we find that the above terms are now proportional to the factor $\overline{K}_2 = \sinh(2r)$ which is a manifestation of the nonstationarity introduced into the system by the squeezing of the bath and goes to zero for the case of no squeezing. All these terms are also proportional to an exponential factor $e^{-p(t-a)}$ which, after a time scale of $t_0 = a + \frac{c_1}{4\gamma_0}$ (a is a constant depending upon the squeezing parameters and c_1 is a constant), drives these terms to zero thereby attaining the usual thermal state. This is the time scale over which the system forgets the initial squeezing of the bath which agrees with [7]. However, this time scale is much greater than the usual time scales of decoherence, thereby demonstrating that squeezing of the reservoir can greatly influence the decoherence properties of the system [14, 15].

5. Discussion and conclusions

In this paper, we enunciated the general framework of canonical transformations with some applications. After a recapitulation of canonical transformations where we set up the criteria

for a transformation to be canonical, we showed the connection between the exponential vectors, the coherent states and the Weyl operators. We also introduced a more general class of Fock space vectors, the ultracoherent vectors. We then set up the unitary ray representations of the group of canonical transformations and applied the unitary operator of the representation to the exponential vector, relegating its action on the ultracoherent vector to the appendix. An important relation showing the connection between the Weyl operator and the homogeneous canonical transformations was also given. Two general classes of canonical transformations, one involving self-adjoint operators and the other involving symmetric Hilbert–Schmidt operators, were discussed and their structure illustrated.

The rotation and squeezing operators, which have many applications in physics, belong to the above two classes. This connection was demonstrated by analysing the single-mode as well as the n-mode squeeze operators which were then shown to be elements of the general group of canonical transformations. Making use of this identification, we used their unitary ray representations on the exponential as well as the ultracoherent vectors.

We then discussed the effect of squeezing of the bath on the decohering properties of the system. The possibility of constructing a squeezed bath based on the generation of squeezed light has been discussed in [37]. First, we took up the case of a bath consisting of a massless bosonic field with the bath reference states being the squeezed vacuum and squeezed thermal states. The reduced density matrix involved the evaluation of a trace which had the Hamiltonian of the van Hove model in it. Provided that the Hamiltonian is semi-bounded superselection rules are induced exactly under the condition that the boson field is infrared divergent, i.e., the vacuum state disappears in the continuum. Depending on the squeezing parameters of the reservoir the decay rate of the quantum coherences can be suppressed or enhanced at intermediate times, but the large-time behaviour and the superselection structure is not affected by squeezing.

We then studied the effect of a squeezed thermal reservoir on the decoherence properties of the system of a harmonic oscillator with the reservoir being a standard harmonic one. We found that squeezing, resulting in the development of correlations between bath modes, can significantly influence the decoherence properties of the system and can slow down the process of decoherence. In addition to the decoherence causing term ($D_{pp}(t)$), we found that the terms governing diffusion in p^2 ($D_{xx}(t)$) and the anomalous diffusion terms ($D_{xp}(t)$, $D_{px}(t)$) are also influenced by squeezing. But in the limit of large times the final state of the system is always the thermal state.

Appendix. Ultracoherent vectors and canonical transformations

The details for the following statements are presented in [17]. The ultracoherent vectors $\exp(\Omega(A) + f)$ with $A \in \mathcal{D}_1$ and $f \in \mathcal{H}$ have been defined in (2). Thereby $\mathcal{D}_1$ is the set of all symmetric Hilbert–Schmidt operators A on a Hilbert space $\mathcal{H}$ with all eigenvalues of $AA^\dagger$ strictly less than one. This convex set of operators is usually called the Siegel disc. The inner product of two ultracoherent vectors is

$$(\exp(\Omega(A) + f) \mid \exp(\Omega(B) + g)) = (\det{}_{\mathcal{H}}(I - A^\dagger B))^{-\frac{1}{2}}$$
$$\times \exp\left\{ \tfrac{1}{2}\langle f^* \mid C f^* \rangle + \langle f^* \mid (I - BA^\dagger)^{-1} g \rangle + \tfrac{1}{2}\langle g \mid Dg \rangle \right\} \tag{A.1}$$

where

$$C = B(I - A^\dagger B)^{-1} = (I - BA^\dagger)^{-1}B, \tag{A.2}$$
$$D = A^\dagger + A^\dagger C A^\dagger = A^\dagger (I - BA^\dagger)^{-1} = (I - A^\dagger B)^{-1}A^\dagger. \tag{A.3}$$

The ultracoherent vector is uniquely determined by its inner product with the exponential vectors, which follows from (A.1) as

$$(\exp z \mid \exp(\Omega(A) + f)) = \exp\left(\tfrac{1}{2}\langle z^* \mid A z^*\rangle + \langle z^* \mid f\rangle\right). \tag{A.4}$$

This antianalytic function $z \in \mathcal{H} \to \exp\left(\tfrac{1}{2}\langle z^* \mid A z^*\rangle + \langle z^* \mid f\rangle\right)$ represents the ultracoherent vector in the Bargmann–Fock picture of the Fock space.

A unitary ray representation $T(G)$ of the group of canonical transformations can be defined on the Fock space by the action of $T(G)$ on ultracoherent vectors

$$T(G) \exp(\Omega(A) + f) = \det|U|^{-\frac{1}{2}} \det(I + V^\dagger U^{\dagger -1} A)^{-\frac{1}{2}}$$
$$\times \exp\left\{\Omega(\zeta(G; A)) + (U^\dagger + AV^\dagger)^{-1} f - \tfrac{1}{2}\langle f | V^\dagger (U^\dagger + AV^\dagger)^{-1} f\rangle\right\} \tag{A.5}$$

where

$$A \to \zeta(G; A) = (U^\dagger + AV^\dagger)^{-1}(V^T + AU^T) \tag{A.6}$$

is the group action on the Siegel disc $\mathcal{D}_1$. Definition (A.5) satisfies the product law $T(G_2)T(G_1) = \omega(G_2, G_1)T(G_2 G_1)$ with a phase factor $\omega(G_2, G_1) \in \mathbb{C}$, $|\omega(G_2, G_1)| = 1$. The restriction of (A.5) to exponential vectors yields equation (27) of section 2. Since transformations on the Fock space are uniquely determined by their action on exponential vectors, one can derive (A.5) from (27) using the inner product formula (A.1).

The matrix element of the Weyl operator between two ultracoherent vectors can be calculated from (13) and (A.1). Here we only give the result for exponential vectors

$$(\exp f \mid W(h) \exp g) = \exp\left((f \mid g) + (f \mid h) - (h \mid g) - \tfrac{1}{2}\|h\|^2\right), \tag{A.7}$$

which includes the vacuum expectation for $f = g = 0$. From equation (A.7) we can deduce equation (9). From equations (13), (A.5) and (A.7) one can easily deduce the important identity (42).

References

[1] Kim Y S and Noz M E 1991 *Phase Space Picture of Quantum Mechanics* (Singapore: World Scientific)
[2] Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
[3] Feynman R P and Vernon F L 1963 *Ann. Phys., NY* **24** 118
[4] Caldeira A O and Leggett A J 1983 *Physica* A **121** 587
[5] Joos E, Zeh H D, Kiefer C, Giulini D, Kupsch J and Stamatescu I O 2003 *Decoherence and the Appearance of a Classical World in Quantum Theory* 2nd edn (Berlin: Springer)
[6] Haake F and Reibold R 1985 *Phys. Rev.* A **32** 2462
[7] Grabert H, Schramm P and Ingold G L 1988 *Phys. Rep.* **168** 115
[8] Banerjee S and Ghosh R 2003 *Phys. Rev.* E **67** 056120
[9] Viola L and Lloyd S 1998 *Phys. Rev.* A **58** 2733
[10] Shor P W 1995 *Phys. Rev.* A **52** 2493
    Steane A M 1995 *Proc. R. Soc.* A **452** 2551
    Knill E and Laflamme R 1997 *Phys. Rev.* A **55** 900
[11] Zanardi P and Rasetti M 1997 *Phys. Rev. Lett.* **79** 3306
    Duan L M and Guo G C 1997 *Phys. Rev. Lett.* **79** 1953
    Lidar D A, Chuang I L and Whaley K B 1998 *Phys. Rev. Lett.* **81** 2594
[12] Vitali D and Tombesi P 2001 *Phys. Rev.* A **65** 012305
[13] Agarwal G S 1999 *Phys. Rev.* A **61** 013809
    Fonseca-Romero K M, Kohler S and Hänggi P 2004 *Chem. Phys.* **296** 307
[14] Kennedy T A B and Walls D F 1988 *Phys. Rev.* A **37** 152
[15] Kim M S and Bužek V 1993 *Phys. Rev.* A **47** 610
[16] Caves C M 1982 *Phys. Rev.* D **26** 1817
    Milburn G J 1984 *J. Phys. A: Math. Gen.* **17** 737
    Ekert A K and Knight P L 1990 *Phys. Rev.* A **42** 487

[17] Kupsch J and Banerjee S 2004 *Preprint* math-ph/0410049
[18] Cerveró J M and Rodríguez A 2002 *Int. J. Theor. Phys.* **41** 503
[19] Slowikowski W 1988 *Adv. Appl. Math.* **9** 377
[20] Berezin F A 1966 *The Method of Second Quantization* (New York: Academic)
[21] Bogoliubov N 1947 *J. Phys. USSR* **11** 23
[22] Bargmann V 1970 Group representations on Hilbert spaces of analytic functions *Analytic Methods in Mathematical Physics* ed R P Gilbert and R G Newton (New York: Gordon and Breach) pp 27–63
[23] Shale D 1962 *Trans. Am. Math. Soc.* **103** 149
[24] Ottesen J T 1995 *Infinite Dimensional Groups and Algebras in Quantum Physics (Lecture Notes in Physics vol M 27)* (Berlin: Springer)
[25] Ma X and Rhodes W 1990 *Phys. Rev.* A **41** 4625
[26] Caves C M and Schumacher B L 1985 *Phys. Rev.* A **31** 3068
       Schumacher B L and Caves C M 1985 *Phys. Rev.* A **31** 3093
[27] Fisher R A, Nieto M M and Sandberg V D 1984 *Phys. Rev.* D **29** 1107
[28] Itzykson C 1967 *Commun. Math. Phys.* **4** 92
[29] Kupsch J 2000 *J. Math. Phys.* **41** 5945
[30] Kupsch J 2004 *Preprint* math-ph/0402055v2
[31] van Hove L 1952 *Physica* **18** 145
[32] Schroer B 1963 *Fortschr. Phys.* **11** 1
[33] Arai A and Hirokawa M 2000 *Rev. Math. Phys.* **12** 1085
[34] Hu B L and Matacz A 1994 *Phys. Rev.* D **49** 6612
[35] Banerjee S 2004 *Physica* A **337** 67
[36] Dekker H 1977 *Phys. Rev.* A **16** 2116
[37] Bužek V, Knight P L and Kudryavtsev I K 1991 *Phys. Rev.* A **44** 1931